Renormalization, vortices, and symmetry-breaking perturbation

two-dimensional planar model

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The classical planar Heisenberg model is studied at low temperatures by means of renormog a series of exact transformations. A numerical study of the Migdal recursion relatio
tions models with short-range isotropic interactions rapidly become equivalent to a simplified
proposed by Villain. A series of exact transformations then allows us to treat the Villain m
at low temperatures. To lowest order in a parameter which becomes exponentially small
temperature, we reproduce results obtained previously by Kosterlitz. We also examine
symmetry-breaking crystalline fields on the isotropic planar model. A numerical study
recursion scheme suggests that these fields (which must occur in real quasi-two-dimension
strongly relevant variables, leading to critical behavior distinct from that found for the
However, a more exact low-temperature treatment of the Villain model shows that hexag
fields eventually become irrelevant at temperatures below the $T_c$ of the isotropic model.
critical behavior should be experimentally accessible in this case. Nonuniversal behavior may
crystalline fields dominate the symmetry breaking. Interesting duality transformations, wh
analysis of symmetry-breaking fields are also discussed.

$$A[\theta] \equiv \frac{-H}{k_B T} = - \sum_{\langle \vec{r}, \vec{r}' \rangle} K \{ 1 - \cos[\theta(\vec{r}) - \theta(\vec{r}')] \}$$

$$+ \sum_{\vec{r}} \sum_{\rho} h_{\rho} \cos[p \theta(\vec{r})],$$

FIG. 1. Phase diagrams in the $h_{\rho} - T$ plane for $\rho = 6$
and $\rho = 4$. The asterisks denote critical points with con-
tinuously variable critical exponents.

renormalization group study of higher order gradients, cosines and vortices...
Eigenpairs of Toeplitz and disordered Toeplitz matrices with a Fisher-Hartwig symbol

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(LPK is deceased as of October 26, 2015)

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Toeplitz matrices have entries that are constant along diagonals. They model directed transport, are at the heart of correlation function calculations of the two-dimensional Ising model, and have applications in quantum information science. We derive their eigenvalues and eigenvectors when the symbol is singular Fisher-Hartwig. We then add diagonal disorder and study the resulting eigenpairs. We find that there is a “bulk” behavior that is well captured by second order perturbation theory of non-Hermitian matrices. The non-perturbative behavior is classified into two classes: Runaways type I leave the complex-valued spectrum and become completely real because of eigenvalue attraction. Runaways type II leave the bulk and move very rapidly in response to perturbations. These have high condition numbers and can be predicted. Localization of the eigenvectors are then quantified using entropies and inverse participation ratios. Eigenvectors corresponding to Runaways type II are most localized (i.e., super-exponential), whereas Runaways type I are less localized than the unperturbed counterparts and have most of their probability mass in the interior with algebraic decays. The results are corroborated by applying free probability theory and various other supporting numerical studies.
Non-Hermitian Localization in 1d Neural Networks

- Non-Hermitian matrices, with complex eigenvalue spectra, arise naturally in simple models of sparse neural networks.
- Striking departures from the conventional wisdom about localization arise in the one-dimensional non-Hermitian random matrices.
- An intricate eigenvalue spectrum controls the spontaneous activity and induced response. Directed rings of neurons lead to a hole centered on in the density of states in the complex plane.
- All states are extended on the rim of this hole, while the states outside the hole are localized.
Visual stimulus $s(t)$ transferred from retinal neurons $\rightarrow$ LGN $\rightarrow$ V1 region of the visual cortex

Spike rate $r(t)$ depends on orientation of bar moving across the visual field

Pathway from the retina through the lateral geniculate nucleus (LGN) to the primary visual cortex

*Dayan and Abbott: Theoretical Neuroscience*
Random matrix models of the brain (H. Sompolinsky, L. Abbott et al.)

- Random neural connections can be formed during development, with many stochastic attachments of axons and dendrites to other neurons.

- Over time, pruning and strengthening/weakening of connections allow neural circuits to "learn" various functions.

- The spectra and eigenfunctions of completely random neural networks with a mixture of inhibitory and excitatory connections, can describe neural activity during the early stages of development.

Sensory inputs, possibly after a processing step, are sent via feed forward couplings into a circular ring of N neurons. Note that $M(1,2)$ and $M(2,1)$ can not only be unequal, but also of opposite sign, if one direction is excitatory and the other inhibitory.

$v_i = \text{firing rate deviation from background of the } i^{th} \text{ neuron in recurrent network}$

$u_j = \text{input firing rate of the } j^{th} \text{ neuron in the input (feed forward) network}$

$$\tau \frac{dv_i}{dt} = -v_i + \tanh \left[ \sum_{j=1}^{N} M_{ij} v_j + h_i \right], \quad h_i = \sum_{j=1}^{N} W_{ij} u_j$$

$$\tau \frac{dv_i}{dt} \approx -v_i + \sum_{j=1}^{N} M_{ij} v_j + h_i \quad \text{(linear approximation)}$$
Non-Hermitian neural networks with random excitatory ($M(i,j) > 0$) and inhibitory ($M(i,j) < 0$) connections

$M = -\sum_{j=1}^{N} \left[ s_j^+ e^g |j\rangle \langle j+1| + s_j^- e^{-g} |j+1\rangle \langle j| \right]$

g provides a systematic clockwise (g > 0) or counterclockwise (g < 0) directional bias

Study eigenvalues and eigenvectors of directed, banded non-Hermitian random matrices

$s_j^+, s_j^- = \pm 1$, indep. random variables;

Set g = 0 for now $\rightarrow$ random sign model of J. Feinberg and A. Zee, PRE 59 6433 (1999)
Result of exact diagonalization of 10,000 $N \times N$ matrices with $N = 5000$ and $g = 0$

What are the localization properties of this spectrum??

 Eigenvector distribution in the complex plane $\lambda = \lambda_1 + i\lambda_2$

What would Leo say...

Fractals – where’s the Physics?? (Physics Today...)
What does “localization” mean?

Eigenfunctions within circle on right side are highly localized w/real eigenvalues.

Eigenfunctions in an annulus closer to the origin are more extended.

Localizes length diverges near the origin:

\[ \xi(\lambda_1, \lambda_2) \approx \text{const.} \frac{1}{(|\lambda_1| + |\lambda_2|)\sqrt{\lambda_1^2 + \lambda_2^2}} \]
What is the effect of the bias parameter $g$?

$$M = \begin{pmatrix}
0 & s_1^+ e^g & 0 & \ldots & s_N^+ e^g \\
\bar{s}_1^+ e^{-g} & 0 & s_2^+ e^g & 0 \\
0 & \bar{s}_2^+ e^{-g} & \ddots & \ddots & 0 \\
\vdots & 0 & \ddots & & s_{N-1}^+ e^g \\
s_{N-1}^+ e^g & \ldots & 0 & \bar{s}_{N-1}^+ e^{-g} & 0
\end{pmatrix}$$

As $g$ increases from 0, it tunes down the amount of back propagation in a “feed clockwise” recurrent network...

$s_j^+ = \pm 1$,  $s_j^- = \pm 1$ with equal probability

$0 \leq g < \infty$ (no Dale's law for now)

Similar layered neural nets used for image classification, etc. in machine learning algorithms.

Many layers $\rightarrow$ “deep learning”
Effect of a directional bias around the chain ($g > 0$)

$N = 5000, \quad g = 0.0$
Effect of a directional bias around the chain ($g > 0$)

$N = 5000, \quad g = 0.002$
Effect of a directional bias around the chain ($g > 0$)

$N = 5000, \ g = 0.01$
Effect of a directional bias around the chain ($g > 0$)

$N = 5000, \ g = 0.05$

A gap or hole appears in the eigenvalue spectrum in the complex plane…
Effect of a directional bias around the chain ($g > 0$)

$N = 5000, \quad g = 0.1$
Effect of a directional bias around the chain ($g > 0$)

$N = 5000, \quad g = 0.2$
Effect of a directional bias around the chain ($g > 0$)

$N = 5000, \quad g = 0.5$
Localization lengths and effect of boundary conditions

Define inverse participation ratio

$$IPR = \sum_j |\phi_j|^4 / \sum_j |\phi_j|^2$$

$$IPR \sim \text{inverse localization length}$$

extended state: $$\phi_j \sim 1 / \sqrt{N}, \forall j$$

$$IPR \equiv \sum_j |\phi_j|^4 / \sum_j |\phi_j|^2 \sim 1 / N << 1$$

localized state, $$\phi_j \sim \exp[-|x_j - x_0| / \xi_{loc}]$$

$$IPR \equiv \sum_j |\phi_j|^4 / \sum_j |\phi_j|^2 = O(1)$$

Eigenvalue spectrum for $$g = 0$$
(or, for any $$g$$ with open boundary conditions!)

- Localization length diverges on the rim of the hole when $$g > 0 \rightarrow$$ extended states

Eigenvalue spectrum for $$g = 0.1$$
with periodic boundary conditions
Large $g$ limit: Plane wave states, all eigenfunctions delocalized

- Trajectories of eigenvalues for $N=100$ and values of $g$ decreasing from 1 down to zero.
- Eigenvalues "flow" in the complex plane.
- Motion stops once eigenvalues localize
Energy gap and rings of extended states also appear for coupled neural clusters.

1000 triangular neural clusters, obeying Dale’s law, and coupled together to form a ring.

Layered neural network with tunable back propagation.

“Band theory” for neural networks?
Thank you!

IN MEMORIAM

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January 24, 1937 – October 26, 2015

Photo: Tom Witten