October 22, 2016

Critical fluctuations:

Leo Kadanoff's legacy

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Leo has been one of the pioneers of two main trends of theoretical physics over the last 50 years

1. THE RENORMALIZATION GROUP

2. CONFORMAL FIELD THEORY

Around 1964 critical phenomena were the subject of an intense activity

M.Fisher (1964)

Correlation Functions and the Critical Region of Simple Fluids

The behavior again appears to be independent of lattice structure.[...] It is remarkable, and perhaps unexpected, that a model as simple as a lattice gas with only nearest neighbor interactions should yield a result for the shape of the coexistence curve so close to the experimental results

B. Widom (1965)

Equation of State in the Neighborhood of the Critical Point

introduces "scaling " : the free energy is an homogeneous fonction of $T - T_c$ and $\rho - \rho_c$.

A. Patashinski, V. Pokrovski ZHETF 50, 439 (1966) **Scale invariance with anomalous dimensions**

UNIVERSALITY

Still in doubt

It seems probable, nonetheless, that the difference of about 0.025 between the experimental and theoretical values of β *is a real discrepancy due, presumably, to the more artificial aspects of the Ising Hamiltonian which, in particular, restricts the molecules to the lattice positions*

Michael Fisher (1964)

REVIEWS OF MODERN PHYSICS, April 1967, Kadanoff et al. : **Static phenomena near critical points: theory and experiment**

The basic theoretical ideas are introduced via the molecular field approach, which brings in the concept of an order parameter and suggests that there are close relations among different phase transition problems. Although this theory is qualitatively correct it is quantitatively wrong, it predicts the wrong values of the critical indices. Another theoretical approach, the "scaling law" concept, which predicts relations among these indices, is described. The experimental evidence for and against the scaling laws is assessed. It is suggested that the scaling laws provide a promising approach to understanding phenomena near the critical point, but that they are by no means proved or disproved by the existing experimental data.

This year we celebrate the 50^{th} anniversary of Leo's fundamental article on BLOCK SPINS

SCALING LAWS FOR ISING MODELS NEAR T_c
Physics Vol. 2, No. 6, pp. 263-272, 1966

Physics Vol. 2, No. 6, pp. 263-272, 1966. Physics Publishing Co.

SCALING LAWS FOR ISING MODELS NEAR T_c*

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(Received 3 February 1966)

Abstract

A model for describing the behavior of Ising models very near T_c is introduced. The description is based upon dividing the Ising model into cells which are microscopically large but much smaller than the coherence length and then using the total magnetization within each cell as a collective variable. The resulting calculation serves as a partial justification for Widom's conjecture about the homogeneity of the free energy and at the same time gives his result $d\nu' = \gamma' + 2\beta$.

1. Introduction

IN a recent paper [1] Widom has discussed the consequences of the assumption that the free energy in a system near a phase transition of second order is a homogeneous function of parameters which describe the deviation from the critical point and has shown that this assumption leads to consequences which roughly agree with our present numerical information [2] about the behavior of such systems. Another paper by Widom [3] written at the same time explores the consequences of an apparently quite independent idea: that the behavior of the interface separating droplets of the "wrong phase" within a system just below a phase transition should be quite similar to the behavior of an interface separating a region of fluctuation in the order parameter from the surrounding medium [4]. Here again information is derived which agrees with numerical calculations and experiment [2].

Widom's ideas about interfaces are based upon physical plausibility arguments; his idea about the homogeneity of the singular part of the free energy is not given any very strong

Block Spins

$a \to 2a$

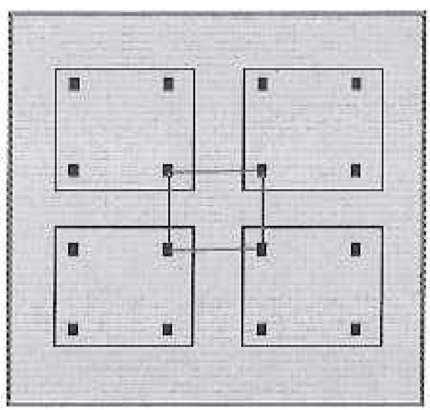
$J \to J'$

$\xi(J) \to \xi(J') = \frac{1}{2}\xi(J)$

$J_c = f(J_c)$

$\xi \sim (J_c - J)^{-\nu}$

$$\nu = \frac{\ln 2}{\ln f'(J_c)}$$



- It introduces the idea of a Hamiltonian flow under a change of the basic lengthscale.

- Critical exponents are related to the flow under rescaling

- Widom's scaling laws are derived

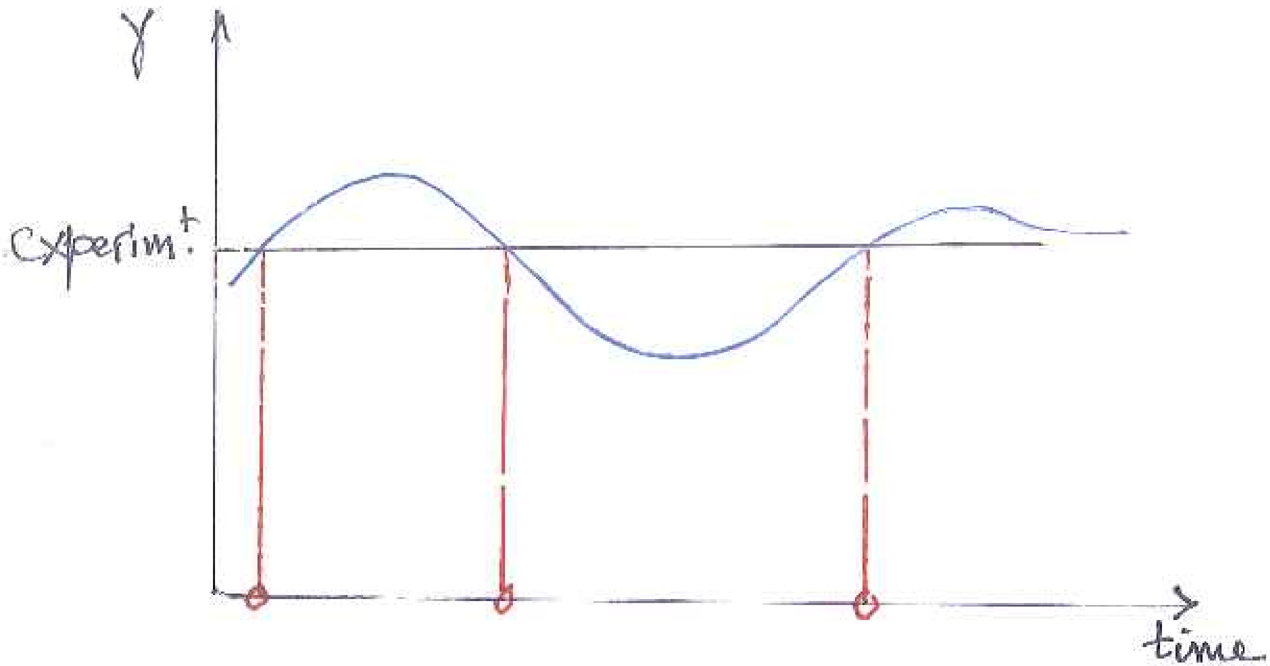
The preprint arrived in France and I read it ; I think I realized how original it was and potentially powerful. But I tried, without any success, to use it to recover the exponents of the 2D Ising model and I gave up. (It's now clear that I was not the only one.)

In 2014 Leo wrote
However the Ising model was the site of my big achievement. Again and again I tried to find y and z, but instead went in circles, making no progress at all.

However as we know this was the gateway to Wilson's renormalization group (Phys.Rev. 1971)
Renormalization group and critical phenomena. I. Renormalization group and the Kadanoff scaling picture

Kadanoff's block spins led to the whole field of real space renormalization group methods. Thousands of papers followed which Leo treated with humor.

Leo's seminar, Harvard 1973



Ken Wilson's insight was remarkable in many ways. In particular it was not clear why he called his picture of Hamiltonian flows, fixed points, ... "renormalization group". The relationship with the 1954 work of Gell-Mann and Low was far from obvious (at least to me).

I was involved in a field theory approach which presented a number of technical advantages. For instance all critical exponents in ϵ or $1/N$ expansions can be computed exactly at T_c , and massless Feynman diagrams are a lot easier.

An unambiguous test of RG

A. I, Larkin and D. E. Khmel'nitskii 1969 (!)
Uniaxial Ferrolectrics with strong dipolar interactions

It contains three major breakthroughs

1. Meanfield theory is quantitatively exact above dimension four (which seeems to have been common knowledge at the Landau Institute, presumably because they had analyzed the consequences of Ginzburg criterion)

2. At d=4 there are calculable logarithmic deviations from mean field theory

For instance the specific heat, instead of having a simple jump at T_c like in mean field theory, should slowly diverge as

$$C = A_{\pm}(|\log|\frac{T - T_c}{T_c}||)^{1/3}$$

with

$$\frac{A_+}{A_-} = \frac{1}{4}$$

Note that this RG prediction is exact, no ϵ-expansion, $1/N$-expansion, real-space RG, etc..

3. If dipolar interactions are not negligible with respect to exchange forces then the four-dimensional theory applies to d=3

Dimension three is accessible to experiments, but after making a crystal with strong dipolar forces, and an easy axis of magnetization (uniaxial means anisotropic crytal)

G.Ahlers, A. Kornblit and H. Guggenheim
Phys.Rev.Lett. 34, 1227 (1975)

LiTbF4 $T_c = 2.885K$

They tested

$$C = A_{\pm}(|\log|\frac{T-T_c}{T_c}||)^{1/3}$$

$$\frac{A_+}{A_-} = \frac{1}{4}$$

and found : The power of the leading logarithmic term is found to be 0.34 ± 0.03, and the corresponding amplitude ratio is 0.24 ± 0.01.

A wonderful test of RG, free of the usual approximate schemes which are not easy to control ... provided the correspondance

$$d = 4 \text{ (short} - \text{range)} \iff \text{d} = 3 \text{ (dipolar)}$$

is true??

Answer : E.B. and J.Zinn-Justin , Phys. Rev B 13., 251(1975)

NO and YES

- NO the correspondance is only qualitative

- YES the correspondance is valid at one-loop level
  ... and almost valid beyond it

- S.R. $d = 4$

$$C = A_{\pm} |\log|t||^{1/3} \left(1 - \frac{25}{81} \frac{\log|\log|t||}{|\log|t||}\right)$$

- Dipolar $d = 3$

$$C = A_{\pm} |\log|t||^{1/3} \left(1 - \frac{1}{243} (108 \log\frac{4}{3} + 41) \frac{\log|\log|t||}{|\log|t||}\right)$$

Corrections are of order

$$O\left(\frac{1}{\log|t|}\right) \text{ with } t = \frac{T - T_c}{T_c}$$

$$25/81 \simeq 0.308$$

$$\frac{1}{243}(108 \log \frac{4}{3} + 41) \simeq 0.296$$

The correspondance is *accidentally* nearly true also at two loop-order.

In practice a $\frac{log(log)}{log}$ is too slowly varying to be measurable, but this still provides one of the best tests of RG since it is free of any approximation.

Beyond the Ising class of universality

Many contributions, among which:

Some critical properties of the eight-vertex model
LP Kadanoff, FJ Wegner - Physical Review B, 1971

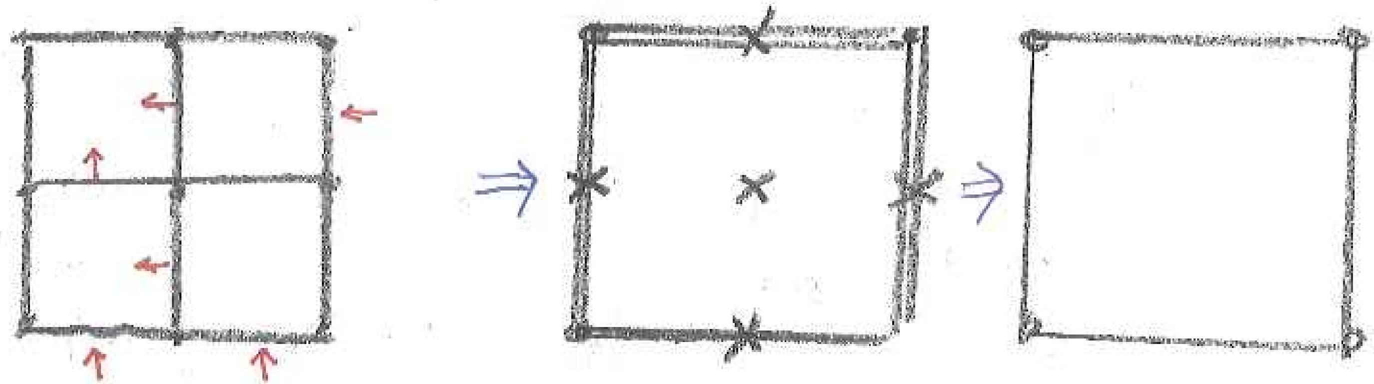
Connections between the critical behavior of the planar model and that of the eight-vertex model
LP Kadanoff - Physical Review Letters, 1977

Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model
JV José, LP Kadanoff, S Kirkpatrick, DR Nelson - Physical Review B, 1977

The Kadanoff-Migdal renormalization group

A.Migdal had produced in 1975 an RG recursion formula which had the advantage of being applicable to the strongly coupled phase of broken internal symmetries or gauge symmetries. Easily extended in dimension d, it allowed one to make contact with the work of Polyakov in 1975 Interaction of Goldstone particles in two dimensions. Applications to ferromagnets and massive Yang-Mills fields. Kadanoff showed that Migdal's RG was simply an ordinary real space decimation-like transformation after a bond-moving approximation, making it easy to implement in many systems, disordered for instance.

Migdal-Kadanoff bond moving RG



1. bond-moving approximation
2. decimation

The operator-product expansion

K. G. Wilson, Phys. Rev. 179, 1499 (1969)
L.P. Kadanoff, Phys. Rev. Lett. 28, 1480 (1969)
OPERATOR ALGEBRA AND THE DETERMINATION OF CRITICAL INDICES

$$O_\alpha(x_1)O_\beta(x_2) = \sum_\gamma c_{\alpha\beta\gamma} O_\gamma(x)$$

$$x = \frac{1}{2}(x_1 + x_2)$$

A pioneering work which had a lot of influence on the development of the consequences of conformal invariance.

Trivial consequences ; for instance the asymptotic behavior of the correlation function

$$< \sigma(r)\sigma(0) >=_{a \ll r \ll \xi} \frac{1}{r^{d-2+\eta}}(A+B(r/\xi)^{1/\nu}+C(r/\xi)^{(1-\alpha)/\nu}+...)$$

(EB D.Amit, Zinn-Justin (1974))

established by M. Fisher and L. Langer, Resistive Anomalies at Magnetic Critical Points (1968).

CONFORMAL SYMMETRY OF CRITICAL FLUCTUATIONS

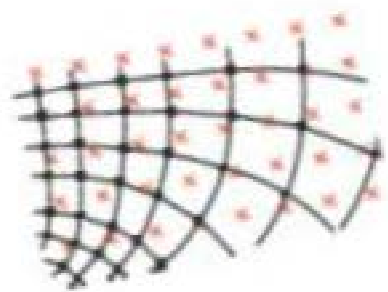
A. M. Polyakov (1970)

Six articles quoted in the bibliography, two refer to Leo's work.

Translation, rotation, dilatation symmetry ,plus

$$x \to x'$$

$$\frac{x'_i}{x'^2} = \frac{x_i}{x^2} + a_i$$



Conformal symmetry at T_c implies

$$< \phi_i(x)\phi_j(y) >= \frac{\delta_{ij}}{(x-y)^{2\Delta_i}}$$

and for three primaries

$$< \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) >= \frac{\lambda_{123}}{(x_{12})^{\Delta_1+\Delta_2-\Delta_3}(x_{23})^{\Delta_2+\Delta_3-\Delta_1}(x_{31})^{\Delta_3+\Delta_1-\Delta_2}}$$

A. M. Polyakov (1970)

But for the four point function, the two projective invariants

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

are undetermined. For instance for a primary field

$$< \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) >= \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} f(u,v)$$

TWO DIMENSIONS

Infinite conformal symmetry of critical fluctuations in two dimensions
A. Belavin, AM Polyakov, AB Zamolodchikov
Journal of Statistical Physics, 1984

Superconformal invariance in two dimensions and the tri-critical Ising model
D Friedan, Z Qiu, S Shenker - Physics Letters B, 1985

The Loewner equation: maps and shapes
IA Gruzberg, LP Kadanoff - Journal of statistical physics, 2004
on the SchrammLoewner evolution (SLE)

$d \geq 3$ The Conformal Bootstrap

Early attempts Polyakov, Migdal ; Ferrara, Grillo, and Gatto (1973)

Used (Vasilev et al.) for four and five loop calculations in perturbative expansions such as ϵ and $1/N$

Developped as a non-perturbative calculational tool by Rychkov et al.

Conformal Bootstrap in Three Dimensions?

Slava Rychkov 2011

Solving the 3D Ising model with the conformal bootstrap

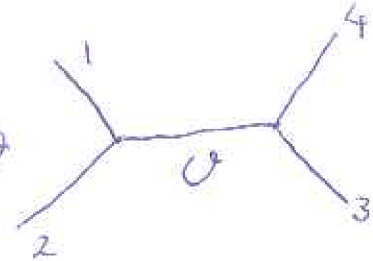
S El-Showk, MF Paulos, D Poland, S Rychkov, 2012

Solving the 3D Ising model with the conformal bootstrap II. c-minimization and precise critical exponents

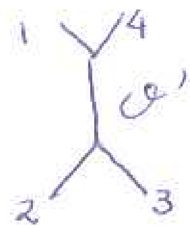
S El-Showk, MF Paulos, D Poland, S Rychkov 2014

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\,\phi_4(x_4)\rangle$$

$$= \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\,\lambda_{34\mathcal{O}}$$



$$= \sum_{\mathcal{O}'} \lambda_{14\mathcal{O}'}\,\lambda_{23\mathcal{O}'}$$



- OPE Associativity (Crossing symmetry)
- Unitarity

These equations do not determine the critical exponents for $d > 2$ ... but they provide a bounded domain of allowed values for Δ_σ and Δ_ϵ (i.e. η and ν) and at present these inequalities give

$$\nu = 0.62999(5) \qquad \eta = 0.03631(3) \qquad \omega = 0.8303(18)$$

a precision which is an order of magnitude above previously known techniques.
However this technology is limited to zero-field and $T = T_c$.
Scaling functions such as the equation of state, correlation functions as functions of r/ξ are, at present, outside the realm of these techniques.

The subject is not yet closed.