Singularities
We have had reunions like this every N years.

They always remind me why we are doing what we are doing.
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They always remind me why we are doing what we are doing.

**How to do science**

**What questions are worth working on**

— Work together: Theory, Simulation and Experiment
— It’s also ok to do theory w/o experiment
— Have courage! You can work on whatever you want. just take it seriously

**The importance of sharing through teaching and otherwise…**

**How to treat each other**
Finite-Time Singularities in the Axisymmetric Three-Dimension Euler Equations

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For pointlike singularities localized well away from the symmetry axis, axisymmetric flows with swirl are arbitrarily well approximated by two-dimensional Boussinesq convection. An adaptive mesh simulation of the latter equations was continued until the maximum three-dimensional vorticity showed a factor of $10^7$ increase, allowing a reasonable determination of exponents, and elucidation of the mechanism of blowup.

The true blowup begins at $t \sim 8.5$. A smooth cap never reforms, the point of maximum $|\nabla \theta|$ is always on or near the symmetry line $x = 0$, and the radius of curvature of the iso-$\theta$ contour near the singularity is larger than but of the order of the thickness (Fig. 3). Analytic estimates [6] suggest that the rollup is not singular. (The mechanism by which a cusp forms on a vortex sheet [10] is not relevant here.) The shape of the singular region is always changing, with new instabilities being born and pushed towards infinity in the rescaled coordinates as the code maintains resolution around the incipient singularity. Since the singularity develops so rapidly, the large, outer scales are effectively frozen. A series of pictures analogous to Fig. 3 for $t \geq 9$ would naturally telescope. At the
A Cascade of Structure in a Drop Falling from a Faucet

X. D. Shi, Michael P. Brenner, Sidney R. Nagel

A drop falling from a faucet is a common example of a mass fissioning into two or more pieces. The shape of the liquid in this situation has been investigated by both experiment and computer simulation. As the viscosity of the liquid is varied, the shape of the drop changes dramatically. Near the point of breakup, viscous drops develop long necks that then spawn a series of smaller necks with ever thinner diameters. Simulations indicate that this repeated formation of necks can proceed ad infinitum whenever a small but finite amount of noise is present in the experiment. In this situation, the dynamical singularity occurring when a drop fissions is characterized by a rough interface.
Iterated Instabilities during Droplet Fission

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Recent observations indicate that the shape of a fluid interface undergoes repeated instabilities arbitrarily close to breakoff. We interpret this behavior as the result of successive instabilities of the similarity solution of Eggers [Phys. Rev. Lett. 71, 3458 (1993)]. We show that the similarity solution is unstable to finite amplitude perturbations, with critical amplitude going to zero at the singularity. Thermal fluctuations in the fluid can trigger the instabilities.
Outline

(1) An Experiment

(2) Vortex Filaments and Biot Savart Singularities (theory)

(4) Iterative Cascade ("theory")
   Scaling arguments and Simulations

(5) Revisiting Vortex Ring Collisions
   (1) Experiments [w/ Shmuel Rubenstein and Ryan McKeown]
   (2) Simulations [w/ Rodolfo Ostilla Monico]
An experiment
An experiment

Instability and reconnection in the head-on collision of two vortex rings

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Head-on Collision of Coloured Vortex Rings at $Re=1071$

Note the formation of small rings from the cross-linking of the wavy vortex filaments of the larger rings.
What is the mechanism for energy to transfer so quickly from large scales to small scales?
Outline

(1) An Experiment

(2) Vortex Filaments and Biot Savart Singularities

(4) Iterative Cascade
   Scaling arguments and Simulations

(5) Revisiting Vortex Ring Collisions
   (1) Experiments [w/ Shmuel Rubenstein and Ryan McKeown]
   (2) Simulations [w/ Rodolfo Ostilla Monico]
Vortex (Ring) Collisions

Start with well separated vortex filaments, with core size much smaller than any other length scale

Track dynamics of collisions

\[ v(r) = -\frac{\Gamma}{4\pi} \int \frac{(r - r_0(s)) \times t(s)}{|r - r_0(s)|^3} \]

\[ \Gamma = \int_c v \cdot dl \]
Vortex (Ring) Collisions

Start with well separated vortex filaments, with core size much smaller than any other length scale

Track dynamics of collisions

\[ \mathbf{v}(\mathbf{r}) = -\frac{\Gamma}{4\pi} \int \frac{\left( \mathbf{r} - \mathbf{r}_0(s) \right) \times \mathbf{t}(s)}{|\mathbf{r} - \mathbf{r}_0(s)|^3} \]

This accurately represents the velocity field as long as:

1. core size << radius of curvature
2. core size << interfilament distance.

\[ \Gamma = \int_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{l} \]
The “inner problem”: inside the core

Adjust core size locally to preserve core volume

\[ A_{\text{core}}(t) = \frac{A_{\text{initial}}}{s_\alpha(t)} \]

Can Show that if:

\[ \frac{dr}{dt} = Wt + Un + Vb, \]

\[ \frac{ds_\alpha}{dt} = \left( \frac{du}{ds} \cdot t \right) s_\alpha = \frac{dW}{ds} s_\alpha - U_\kappa s_\alpha. \]
Vorticity Amplification

\[ \omega(t) = \frac{\Gamma}{A_{core}(t)} = \omega_0 s_\alpha(t) \]

Hence, vorticity blowup directly connected to stretching of filament.
Outer problem: Filament Shape

\[ v(r_0) = -\frac{\Gamma}{4\pi} \log \left( \frac{r_c}{\sigma} \right) \kappa b - \frac{\Gamma}{4\pi} \int' \frac{(r_0 - r(s)) \times t(s)}{|r_0 - r(s)|^3} \, ds. \]

Inner problem: Filament Radius

\[ \sigma^2 = \frac{\sigma_0^2}{s_\alpha}. \]

\[ \frac{ds_\alpha}{dt} = \left( \frac{dv}{ds} \cdot t \right) s_\alpha = \frac{dW}{ds} s_\alpha - U\kappa s_\alpha. \]

Weak coupling of outer & inner due to logarithmic singularity in Biot-Savart
Two qualitative behaviors of Vortex Collisions:

(i) weak interactions — no collision
(ii) direct collision — large vorticity amplification/ large filament curvature. etc.
A Collision

Curvature
Singularity arises from collaboration between “smoke ring” Effect and “point vortex” interaction

$$\frac{\Gamma}{4\pi} \kappa \sim \frac{\Gamma}{2\pi} r^{-1}$$

Dimensional Analysis

$$l(t) = \sqrt{\Gamma (t^* - t)}$$
Similarity Solution

\[ r(s, t) = l(t)G(\eta) \quad \eta = s/l(t) \]

\[ G''_1 = -\frac{1}{\alpha} G'_1 \times \left( G_1 - \sqrt{\Gamma} \int \frac{(G_1 - G_2) \times G'_2}{|G_1 - G_2|^3} d\eta_2 \right) \]
Similarity Solution

\[ r(s, t) = l(t)G(\eta) \quad \eta = \frac{s}{l(t)} \]

\[ \alpha = \ln \left( \frac{R_c}{\sigma} \right) \]

\[ G'' = -\frac{1}{\alpha} G'_1 \times \left( G_1 - \sqrt{\Gamma} \int \frac{(G_1 - G_2) \times G'_2}{|G_1 - G_2|^3} d\eta_2 \right) \]
Filaments become nearly parallel

\[ G_{1}'' = -\frac{1}{\alpha} G_{1}' \times \left( G_{1} - \sqrt{\Gamma} \int \frac{(G_{1} - G_{2}) \times G_{2}'}{|G_{1} - G_{2}|^{3}} \, d\eta_{2} \right) \]

\[ \rightarrow 0 \text{ as } \alpha \rightarrow \infty \]
Localization of integral converts it to a set of delay differential equations.

\[ G''_1 = -\frac{1}{\alpha} G'_1 \times \left( G_1 - \sqrt{\Gamma} \int \frac{(G_1 - G_2) \times G'_2}{|G_1 - G_2|^3} \, d\eta_2 \right) \]

\[ G''_1 = -\frac{1}{\alpha} G'_1 \times \left( G_1 - \frac{(G_1 - G_2(\eta_2)) \times G'_2(\eta_2)}{|G_1 - G_2(\eta_2)|^2} \right) \]

Converts it to a set of delay differential equations.
Similarity Solutions are Double Tents

Two parameters:

D: closest approach of filaments

chi: angle between tangent vectors at closest approach
Inner Solution: What happens to the core radius?

Only decreases by ~ factor of 2. Violates asymptotic assumption.
Core Radius Always Decreases too Slowly to Maintain Vortex Approximation

This occurs because of the logarithmic divergence of Biot-Savart law

(Paper — JFM 2012) — this result generalizes to collision of multiple vortex filaments with arbitrary circulations.
Double Tents have long been observed.....
Double Tents have long been observed.....

Numerical Study of Vortex Reconnection

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Cross-linking of two antiparallel vortex tubes

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FIG. 2. Wire plots of $|\omega|$ surfaces, all from the same view angle: (a) $t = 0$; (b) $t = 1$; (c) $t = 2$; (d) $t = 3$; (e) $t = 3.75$; (f) $t = 4.5$; and (g) $t = 6$. 
Direct simulations of Biot-Savart (fixed core)
What happens next?

Core Radius always decreases too slowly to keep up

Flattening of the vortex core
We can use the similarity solution to compute how filaments convert into sheets.

\[ a = f(r, R) \]
\[ b = g(r, R) \]
Vortex collisions lead to two finite thickness vortex sheets

The aspect ratio can be strikingly small:

\[
\frac{a}{b} = \left( \frac{R_0}{r_0} \right)^{\frac{2(\alpha + \beta)}{1+2\beta}}
\]

\[
\alpha, \beta = \frac{1}{2}
\]

\[
\frac{a}{b} = \frac{R_0}{r_0}
\]

\(\alpha, \beta\) depends on prefactors of similarity solution
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   (1) Experiments [w/ Shmuel Rubenstein and Ryan McKeown]
   (2) Simulations [w/ Rodolfo Ostilla Monico]
An iterative Possibility
Finite thickness vortex sheets are unstable

\[ u^2 = \frac{V^2}{b^2} \left\{ (kb - 1)^2 - e^{-\frac{2}{\kappa b}} \right\} \]

Most unstable wavelength = 8*b
The iterative cascade

Can this occur indefinitely? Can it result in a singularity?
The iterative cascade

Can this occur indefinitely?
Can it result in a singularity?

\[ \Gamma_{\text{new}} = \frac{\Gamma}{ab} 8b^2 \]

\[ \pi r_{\text{new}}^2 = 8b^2 \]
The iterative cascade

After an iteration:

\[ \Gamma_{n+1} = \frac{\Gamma}{a_n b_n} (8b_n^2) = 8\Gamma_n \left(\frac{b_n}{a_n}\right) \]

\[ \pi r_{n+1}^2 = 8r_n^2 \left(\frac{r_n}{R_n}\right)^{\frac{4\alpha}{1+2\beta}} \]

So:

\[ \frac{r_{n+1}^2}{R_{n+1}^2} = \frac{8}{\pi} \left(\frac{r_n}{R_n}\right)^{\frac{4(\alpha+\beta)}{1+2\beta}} \]

\[ \Gamma_{n+1} = 8\Gamma_n \left(\frac{r_n}{R_n}\right)^{\frac{2(\alpha+\beta)}{1+2\beta}} \]

Vorticity conservation

area conservation

A map for relating nth step to n+1st step
Fixed point for \( r/R \)

If \( \alpha > \frac{1}{2} \)

\[
\frac{r_\infty}{R_\infty} = \left( \frac{\pi}{8} \right)^{\frac{1+2\beta}{4\alpha-2}}.
\]

\( \Gamma_n \to 0 \)

**Remarks:**

- The cascade occurs in finite time
- Fixed point of map is unstable
Outline

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Syringe Pump

Synchronize
- Galvanometer
- Pump
- Camera
- Pulsed Laser

Scanning Laser Sheet
80-20 Base

Vortex Cannons

High-speed Camera

Pulsed Laser

Syringe Pump
Experimental Video
Flattened Vortex Core!
3D Rendering Technique

Scan at $t_j$

$z_n \rightarrow z_1$

$I(x, y, z, t)$

Reconstruct 2D Slices into 3D Volumes

Array of 3D Volumes

$t_m \rightarrow t_1$

$I(x, y, z, t)$
View Explanation

Dyed Vortex Ring

Un-Dyed Vortex Ring

Back View

Side View

Front View
Vortex Core Breakdown

Back View

Side View

Front View
Flattened Core..
Is the transition to “turbulent smoke” a consequence of a Biot Savart Singularity?

Using code: AfID
— Direct numerical simulation of Navier Stokes Equations
— 2nd order (energy conserving) finite differences
— Cylindrical coordinates, conserves symmetries
Start with 2 antiparallel vortex rings

Seed with small (most unstable) perturbation \([N=5]\)

\[
R_2(\theta) = R_0 + \epsilon \cos(\lambda[\theta + \varphi_0])
\]

\[
R_1(\theta) = R_0 + \epsilon \cos(\lambda\theta)
\]

w/o perturbation, form growing vortex dipole
Biot Savart Simulation \( \epsilon = 0.05 \phi = 180^\circ \)
Navier Stokes Simulations:
double tents

pressure fields
Navier Stokes Simulations:
Smoke product and double tents

pressure fields
The production of small scales in vortex collisions in real flows (vortex ring collision) are preceded by “Biot Savart singularities”, intermediate asymptotic solutions to the Euler equations. These solutions can produce very small scales, very quickly: (Biot Savart-Singularity + “Neu” vortex flattening)

Sheet thickness $\sim r_0/R_0 \times \text{initial scales}$
Timescale $\sim R_0^2/\Gamma$

\[ b_n = r_n \left( \frac{R_n}{r_n} \right)^{-\frac{2\alpha}{1+2\beta}} \]
Summary

Simulations very rapidly run out of resolution.

Numerical simulations performed to date probe large $r_0/R_0$ and might miss interesting phenomena.

Experiments: Events happen too quickly and at too small of spatial scales to be resolved.

Current experimental advances (Rubenstein/McKeown) show that the phenomenology is quite rich and needs to be resolved.
Summary

All of this has dramatically changed my views on the potential existence of an Euler singularity.

We just don't have the numerical resolution to rule out the scenarios that I've outlined, or discover others.

Even the largest super computers can't see more than an iteration or so (w/o rampant remeshing).

Experimental situation slightly better—but not much so.
  Laser sheet: 1400 scans/sec in 3d (25 slices/scan)
  Spatial resolution: 140 microns/voxel
Vortex Ring Experiments & Simulations

Shmuel Rubenstein

Ryan McKeown

Rodolfo Ostilla Monico

Sahand Hormuz

Alain Pumir